Shore

Student Number:

Set:

Year 12 Mathematics Trial Examination 2010

General Instructions

- Reading time − 5 minutes
- Working time − 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided on the back page of this question paper
- All necessary working should be shown in every question

Note: Any time you have remaining should be spent revising your answers.

Total marks − 120

- Attempt Questions 1 − 10
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Total Marks − 120
Attempt Questions 1−10
All questions are of equal value

Answer each question is a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) **Marks**

(a) Evaluate e^{-3} correct to 3 significant figures. **2**

(b) Factorise $8x^3 - 125$. **2**

(c) Simplify $\dfrac{5x-3}{x^2-9} - \dfrac{2}{x-3}$. **2**

(d) Find the values of x for which $|x+1| \le 4$. **2**

(e) Find the integers a and b such that $\left(5-\sqrt{2}\right)^2 = a - b\sqrt{2}$. **2**

(f) Calculate the limiting sum of the geometric series $\dfrac{5}{6} + \dfrac{5}{36} + \dfrac{5}{216} + \ldots$. **2**

Question 2 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Differentiate with respect to x:

(i) $\left(x^3+7\right)^4$ **2**

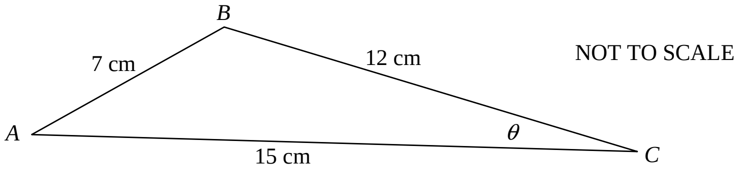
(ii) $x\sin x$ **2**

(iii) $\dfrac{e^x}{2x+1}$ **2**

(b) Find $\int\left(\sec^2 3x + x\right)dx$. **2**

(c) Evaluate $\int_0^1 \dfrac{dx}{x+2}$. **2**

(d) **2**

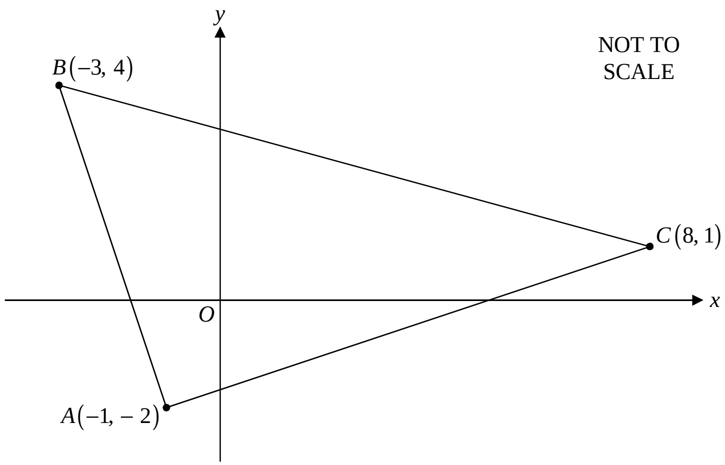


The diagram shows ΔABC with $\angle ACB = \theta$, $AB = 7$ centimetres, $BC = 12$ centimetres and $AC = 15$ centimetres.

Find the value of θ correct to the nearest degree.

Question 3 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



The diagram shows the points $A(-1,-2)$, $B(-3,4)$ and $C(8,1)$.

(i) Find the gradient of AB. **1**

(ii) Show that AB is perpendicular to AC. **2**

(iii) Find the length of the interval AC. **1**

(iv) Hence, or otherwise, calculate the area of the triangle ABC. **2**

(b) Find the equation of the tangent to the curve $y = 3e^{2x}$ at the point on the curve where $x = \frac{1}{2}$. **3**

(c) Let α and β be the solutions of $x^2 - 3x + 7 = 0$.

(i) Find $\alpha\beta$. **1**

(ii) Find $\alpha + \beta$. **1**

(iii) Hence, find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet **Marks**

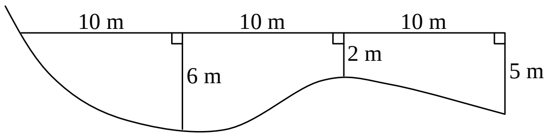
(a) Find the values of k for which the quadratic equation $5x^2 - 2x + k = 0$ has no real roots. **2**

(b) Four red marbles and five green marbles are contained in a cloth bag. Two marbles are randomly selected without replacement.

(i) Find the probability of selecting two marbles of the same colour. **2**

(ii) Find the probability of selecting two marbles of different colours. **1**

(c) The diagram below shows the cross-section of a river with the depths of the water shown in metres, at 10 metre intervals.



(i) Use the trapezoidal rule to find an approximate value for the area of the cross-section. **2**

(ii) Water flows through this section of the river at a speed of 0.6 metres per second. **2**

Calculate the approximate volume of water that flows through this cross-section in one hour.

(d) Consider the parabola $8y = x^2 - 6x - 23$.

(i) Find the coordinates of the vertex. **2**

(ii) Find the coordinates of the focus. **1**

Question 5 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) State the domain of the function $y = \sqrt{36 - x^2}$. **1**

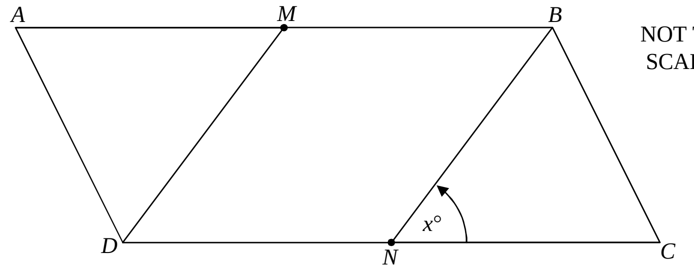
(b) Fred is training for a big running race. On the first day he runs 5 kilometres. On each subsequent day he runs 200 metres further than he did on the previous day. He stops training on the day he runs 42.2 kilometres.

(i) How far does Fred run on the 50^{th} day? **2**

(ii) How many days does Fred train for? **1**

(iii) What is the total distance that Fred runs during his training? **2**

(c)



In the diagram, $DMBN$ is a rhombus. M and N are the midpoints of AB and CD respectively and $\angle CNB = x°$.

Copy or trace the diagram into your writing booklet.

(i) Show that $\angle AMD = x°$, giving reasons. **2**

(ii) Prove that $\Delta AMD \equiv \Delta CNB$. **3**

(iii) Prove that $ABCD$ is a parallelogram. **1**

Question 6 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) Solve $2\sin^2 x - 7\sin x + 3 = 0$ for $0 \le x \le 2\pi$. **3**

(b) (i) Find $\frac{d}{dx}\left[\log_e(\sin 2x)\right]$. **2**

(ii) Hence, or otherwise, evaluate $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cot 2x \, dx$. **2**

(c) Consider the function $f(x) = x^3 - 3x^2 + 8$.

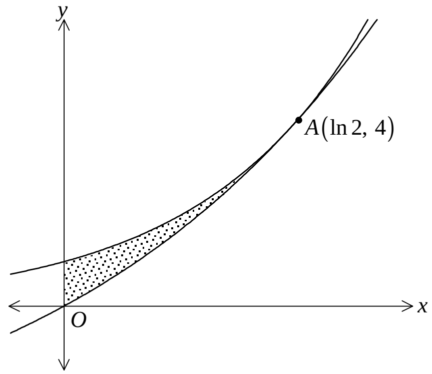
(i) Find the coordinates of the stationary points of the function $y = f(x)$, and determine their nature. **3**

(ii) Sketch the function $y = f(x)$, showing the stationary points. **1**

(iii) Find the values of x for which the curve $y = f(x)$ is concave down. **1**

Question 7 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) **3**



NOT TO SCALE

The graphs of the curves $y = e^{2x}$ and $y = 4e^x - 4$ are shown in the diagram above. The curves intersect at the point $A(\ln 2, 4)$.

Calculate the exact area of the shaded region.

(b) A tank initially holds 2500 litres of water. The water drains from the bottom of the tank. The tank takes 50 minutes to empty.

A mathematical model predicts that the volume, V litres, of water that will remain in the tank after t minutes is given by

$$V = 2500\left(1 - \frac{t}{50}\right)^2, \text{ where } 0 \le t \le 50.$$

(i) What volume does the model predict will remain after 10 minutes? **1**

(ii) At what rate does the model predict that the water will drain from the tank after 20 minutes? **2**

(iii) At what time does the model predict that the water will drain at its fastest rate from the tank? **2**

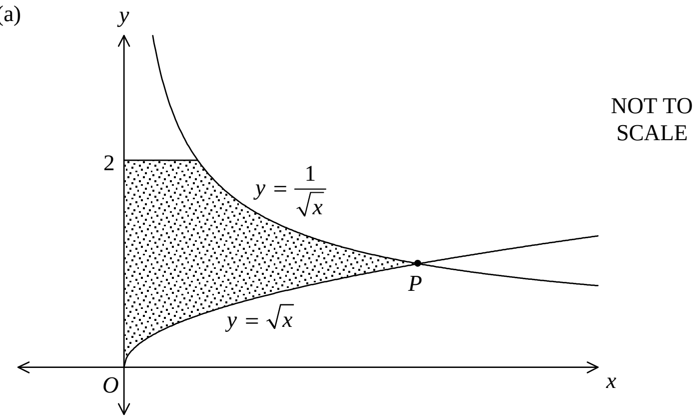
(c) A superannuation fund pays interest at the rate of 5% per annum compounding annually. Steven decides to invest \$7000 into the fund at the beginning of each year, commencing on the 1st of January 2011.

(i) Write an expression for the value of Steven's fund after 3 years. **1**

(ii) What will be the value of Steven's superannuation when he retires on the 31st of December 2041? **3**

Question 8 (12 marks) Use a SEPARATE writing booklet **Marks**

(a)



The graphs of the curves $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$ intersect at the point P, as shown in the diagram above.

(i) Show that P is the point (1, 1). **1**

(ii) Find the area of the shaded region bounded by $y = \sqrt{x}$, $y = \dfrac{1}{\sqrt{x}}$, the y-axis and the line $y = 2$. **3**

(b) Solve $\log_7 x - \log_7 4 = 2\log_7 3$. **2**

(c) The velocity of a particle is given by $v = 3 - 6\cos t$ for $0 \le t \le 2\pi$, where v is measured in metres per second and t is the time in seconds.

(i) Sketch the graph of v as a function of t for $0 \le t \le 2\pi$. **2**

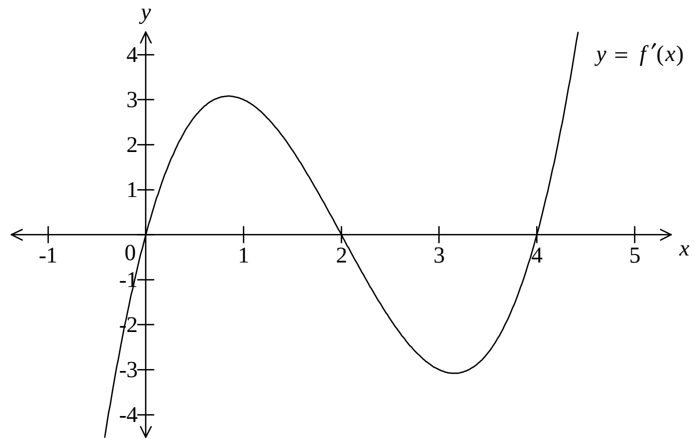
(ii) At what times during this period is the particle at rest? **2**

(iii) Find an expression for the acceleration, a m/s^2, in terms of t. **1**

(iv) Find when the particle first reaches its maximum acceleration. **1**

Question 9 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) **2**



The above diagram shows a sketch of the gradient function of the curve $y = f(x)$.

In your writing booklet, draw a possible sketch of the function $y = f(x)$ given that $f(0) = 1$.

(b) The radioisotope Technetium-99m is used for medical procedures and is produced at Lucas Heights in NSW. Technetium-99m has a rate of decay that is proportional to the mass M present at any given time, such that $\dfrac{dM}{dt} = -kM$.

(i) Show that $M = M_0 e^{-kt}$, where k and M_0 are constants, satisfies the differential equation above. **1**

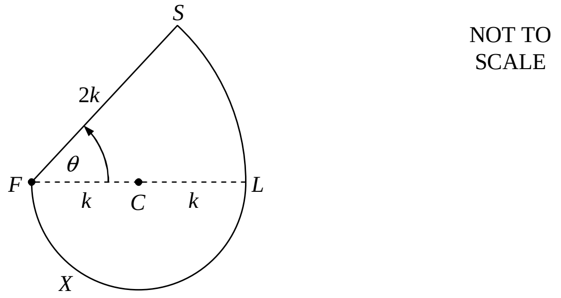
(ii) Technetium-99m has a half life of 6 hours. That is, the time taken for half the initial mass to decay is 6 hours. Find the value of k. **2**

(iii) A sample of Technetium-99m was shipped from the production site to a hospital in Western Australia. The total shipping time was 15.6 hours. **2**

How many kilograms were shipped if just **one** kilogram of Technetium-99m arrived at the hospital?

Question 9 continues

(c)



A cam is formed with cross-section as shown in the diagram. The cross-section consists of a semi-circle *FLX*, with centre *C* and radius *k*, and a sector *FSL*, with centre *F*, radius 2*k* and angle θ radians.

(i) Show that the perimeter *P* of the cam is given by $P = k\left(2\theta + \pi + 2\right)$. **1**

(ii) The area of the cross-section is 1 unit^2. Find an expression for θ in terms of *k*. **2**

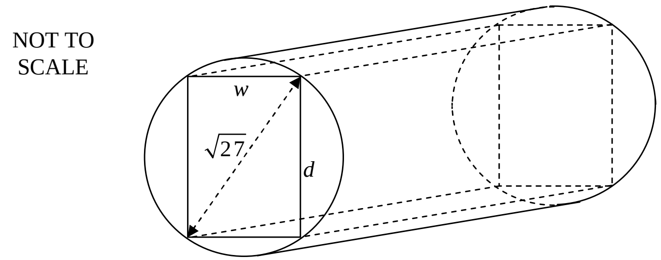
(iii) Hence, show that the perimeter *P* is given by **2**

$$P = \frac{1}{k} + k\left(2 + \frac{\pi}{2}\right).$$

End of Question 9

Question 10 (12 marks) Use a SEPARATE writing booklet **Marks**

(a) A rectangular beam of width *w* cm and depth *d* cm can be cut from a cylindrical log of wood as shown in the diagram below.



The diameter of the cross-section of the log (and hence the diagonal of the cross-section of the beam) is $\sqrt{27}$ cm.

The strength *S* of the beam is proportional to the product of its width and the square of its depth, so that $S = kd^2w$, where *k* is a positive constant.

(i) Show that $S = k(27w - w^3)$. **2**

(ii) What numerical dimensions will give a beam of maximum strength? Leave your answer as an exact value. **3**

(iii) A square beam with diagonal of $\sqrt{27}$ cm is to be cut from an identical log. Show that the rectangular beam of maximum strength is more than 8% stronger than this square beam. **2**

(b) Consider the function $f(x) = x\left((\ln x)^2 - 2\ln x + 2\right)$.

(i) Show that $f'(x) = (\ln x)^2$. **3**

(ii) Hence, or otherwise, find the volume of the solid of revolution formed when the region bounded by the curve $y = \ln x$ and the *x*-axis between $x = 1$ and $x = e$ is rotated about the *x*-axis. **2**

END OF EXAM

Question 1

a) $e^{-3} = 0.049787\ldots$

$\doteqdot 0.0498$

b) $8x^3 - 125 = (2x)^3 - (5)^3$

$= (2x-5)\left((2x)^2 + 2x \times 5 + 5^2\right)$

$= (2x-5)(4x^2 + 10x + 25)$

c) $\dfrac{5x-3}{(x^2-9)} - \dfrac{2}{x-3} = \dfrac{5x-3}{(x+3)(x-3)} - \dfrac{2(x+3)}{(x+3)(x-3)}$

$= \dfrac{5x - 2x - 3 - 6}{(x+3)(x-3)}$

$= \dfrac{3x - 9}{(x+3)(x-3)}$

$= \dfrac{3(x-3)}{(x+3)(x-3)}$

$= \dfrac{3}{(x+3)}$

d) $|x+1| \leq 4 = -4 \leq (x+1) \leq 4$

$-5 \leq x \leq 3$

e) $(5-\sqrt{2})^2 = (5-\sqrt{2})(5-\sqrt{2})$

$= 25 - 10\sqrt{2} + 2$

$= 27 - 10\sqrt{2}$

$\therefore a = 27$

$b = 10$

f) $a = \frac{5}{6}$ $r = \frac{1}{6}$ $S_\infty = \dfrac{a}{1-r} = \dfrac{\frac{5}{6}}{1-\frac{1}{6}} = \dfrac{5/6}{5/6} = 1$

Question 2

a) i) $\dfrac{d}{dx}(x^3+7)^4 = 4(x^3+7)^3 \times 3x^2$

$= 12x^2(x^3+7)^3$

ii) $\dfrac{d}{dx}(x \sin x) = x \cos x + 1 \times \sin x$

$= x \cos x + \sin x$

$u = x$ $v = \sin x$

$u' = 1$ $v' = \cos x$

iii) $\dfrac{d}{dx}\left(\dfrac{e^x}{2x+1}\right) = \dfrac{(2x+1)e^x - 2e^x}{(2x+1)^2}$

$u = e^x$ $v = 2x+1$

$u' = e^x$ $v' = 2$

$= \dfrac{e^x(2x-1)}{(2x+1)^2}$

b) $\int (\sec^2 3x + x)\,dx = \frac{1}{3}\tan 3x + \dfrac{x^2}{2} + c$

c) $\int_0^1 \dfrac{dx}{x+2} = \left[\ln(x+2)\right]_0^1$

$= \ln 3 - \ln 2$

$= \ln \frac{3}{2}$

d) $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{12^2 + 15^2 - 7^2}{2 \times 12 \times 15}$

$= \dfrac{320}{360}$

$\theta = \cos^{-1}\left(\dfrac{320}{360}\right)$

$= 27.2660 \doteqdot 27^\circ$

Question 3

a) i) $m_{AB} = \frac{4-(-2)}{-3-(-1)}$

$= \frac{6}{-2}$

$= -3$

ii) $m_{AC} = \frac{1-(-2)}{8-(-1)}$

$= \frac{3}{9}$

$= \frac{1}{3}$

$m_{AC} \times m_{AB} = -3 \times \frac{1}{3}$

$= -1$

$\therefore AC \perp AB$

iii) $d_{AC} = \sqrt{(8-(-1))^2 + (1-(-2))^2}$

$= \sqrt{9^2 + 3^2}$

$= \sqrt{81+9}$

$= \sqrt{90}$

iv) $d_{AB} = \sqrt{(-3-(-1))^2 + (4-(-2))^2}$

$= \sqrt{4 + 36}$

$= \sqrt{40}$

$\text{Area}_{\Delta ABC} = \frac{1}{2} \times \sqrt{90} \times \sqrt{40}$

$= 30 \text{ units}^2$

Question 3 continued

b) $y = 3e^{2x}$

$y' = 6e^{2x}$

when $x = \frac{1}{2}$

$y = 3e^{2\times\frac{1}{2}}$ $\quad y' = 6e^{2\times\frac{1}{2}}$

$= 3e$ $\quad = 6e$

The eqn of a straight line is given by;

$y - y_1 = m(x - x_1)$

$y - 3e = 6e(x - \frac{1}{2})$

$y - 3e = 6ex - 3e$

$y = 6ex$

c) $x^2 - 3x + 7 = 0$

i) $\alpha\beta = \frac{c}{a}$

$= \frac{7}{1}$

$= 7$

ii) $\alpha + \beta = \frac{-b}{a}$

$= \frac{--3}{1}$

$= 3$

iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{3}{7}$

Question 4

a) $5x^2 - 2x + k = 0$

For no real roots $\Delta < 0$

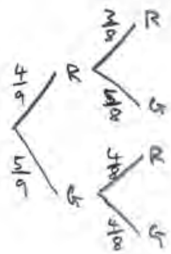
$b^2 - 4ac < 0$

$(-2)^2 - 4 \times 5 \times k < 0$

$4 - 20k < 0$

$-20k < -4$

$k > \frac{1}{5}$



b) i) $P_{(\text{same colour})} = P_{(RR)} + P_{(GG)}$

$= \left(\frac{4}{9} \times \frac{3}{8}\right) + \left(\frac{5}{9} \times \frac{4}{8}\right)$

$= \frac{12}{72} + \frac{20}{72}$

$= \frac{32}{72}$

$= \frac{4}{9}$

ii) P (different colour) $= 1 - P$(same colour)

$= 1 - \frac{4}{9}$

$= \frac{5}{9}$

Question 4 continued

c) i) Area $\doteqdot \frac{h}{2}\left[(h_f + h_e) + 2(\text{middles})\right]$

$= \frac{10}{2}\left[(0 + 5) + 2(6 + 2)\right]$

$= 5(5 + 16)$

$= 105\,m^2$

ii) Volume $= 105\,m^2 \times 0.6\,ms^{-1} \times 3600$

$= 226800\,m^3/h$

$\therefore$ Approx $226800\,m^3$ flow through this cross-section in 1 hour.

d) i) $8y = x^2 - 6x - 23$

$8y + 23 = x^2 - 6x$

$8y + 32 = x^2 - 6x + 9$

$8(y + 4) = (x - 3)^2$ $\qquad 4a(y - k) = (x - h)^2$

$\therefore$ Vertex at $(3, -4)$

ii) $4a = 8$

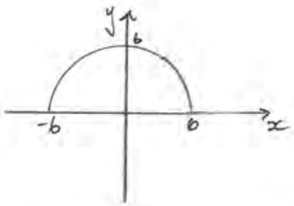
$a = 2$

$\therefore$ focal length $= 2$ and parabola is concave up

$\therefore$ focus at $(3, -2)$

Question 5

a) $y = \sqrt{36 - x^2}$



Domain is $-6 \leq x \leq 6$

b) $a = 5 \quad d = 0.2 \quad l = 42.2$

i) $T_n = a + (n-1)d$

$T_{50} = 5 + (50-1)0.2$

$T_{50} = 5 + 49 \times 0.2$

$= 14.8$ km

ii) $42.2 = 5 + (n-1)0.2$

$\frac{37.2}{0.2} = n - 1$

$n = \frac{37.2}{0.2} + 1$

$= 187$

$\therefore$ Fred trains for 187 days.

iii) $S_n = \frac{n}{2}(a + l)$

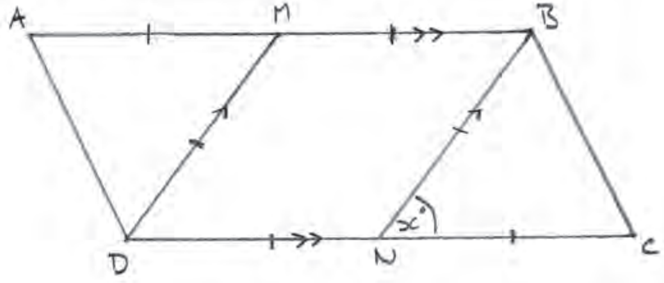
$= \frac{187}{2}(5 + 42.2)$

$= 4413.2$

$\therefore$ Fred runs a total distance of 4413.2 km.

Question 5 continued

c)



i) $\angle MBN = x°$ (alternate angles on // lines)

$\therefore \angle AMD = x°$ (corresponding angles on // lines)

ii) In ΔAMD and ΔCNB

$DM = MB = BN = DN$ (sides of a rhombus)

$AM = MB$ (Given)

$NC = DN$ (Given)

$\therefore AM = NC$ (S)

$\angle AMD = \angle CNB$ (Proved in part (i)) (A)

$DM = BN$ (sides of a rhombus) (S)

$\therefore \Delta AMD \equiv \Delta CNB$

iii) $AB = 2AM$ (M is midpoint of AB)

$CD = 2NC$ (N is midpoint of CD)

Since $AM = NC$ (proved in part (ii))

$\therefore AB = CD$

$AD = CB$ (matching sides of congruent triangles)

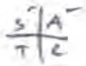
$\therefore$ ABCD is a parallelogram
(two pairs of equal opposite sides)

Question 6

a) $2\sin^2 x - 7\sin x + 3 = 0$ let $u = \sin x$

$$2u^2 - 7u + 3 = 0$$
$$(2u - 1)(u - 3) = 0$$

$\therefore u = 3$ or $u = \frac{1}{2}$

$\therefore \sin x = 3$ or $\sin x = \frac{1}{2}$  

No soln as $|\sin x| \leq 1$ for all x

$\therefore x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$

$= \frac{\pi}{6}$ or $\frac{5\pi}{6}$

b) i) $\frac{d}{dx}\left[\log_e(\sin 2x)\right] = \frac{2\cos 2x}{\sin 2x}$

$= 2\cot 2x$

ii) $\int_{\pi/8}^{\pi/4} \cot 2x \, dx = \frac{1}{2}\int_{\pi/8}^{\pi/4} 2\cot 2x \, dx$

$= \frac{1}{2}\left[\log_e(\sin 2x)\right]_{\pi/8}^{\pi/4}$

$= \frac{1}{2}\left[\log_e(\sin 2.\frac{\pi}{4}) - \log_e(\sin 2.\frac{\pi}{8})\right]$

$= \frac{1}{2}\left[\ln 1 - \ln \frac{1}{\sqrt{2}}\right]$

$= \frac{1}{2}\left[0 - (\ln 1 - \ln\sqrt{2})\right]$

$= \frac{1}{2}(0 - 0 + \ln\sqrt{2})$

$= \frac{1}{2}\ln\sqrt{2}$

$= \frac{1}{4}\ln 2$

Question 6 continued

c) $f(x) = x^3 - 3x^2 + 8$

i) $f'(x) = 3x^2 - 6x$

$= 3x(x - 2)$

Stationary points when $f'(x) = 0$

$\therefore 0 = 3x(x-2)$

$x = 0$ and $x = 2$

$f(0) = 0^3 - 3(0)^2 + 8$
$= 8$

$f(2) = 2^3 - 3.2^2 + 8$
$= 8 - 12 + 8$
$= 4$

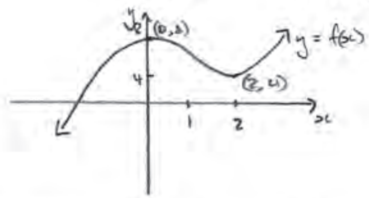
Stat pts at $(0,8)$ and $(2,4)$

$f''(x) = 6x - 6$

$f''(0) = 6.0 - 6$
$= -6 < 0$ $\therefore$ concave down at $(0,8)$
$\therefore (0,8)$ is a maximum.

$f''(2) = 6.2 - 6$ $\therefore$ concave up at $(2,4)$
$= 6 > 0$ $\therefore (2,4)$ is a minimum.

ii)



iii) concave down when $f''(x) < 0$

$6x - 6 < 0$ $\therefore f(x)$ is concave down
$x < 1$ for $x < 1$

Question 7

a) $$\text{Area} = \int_0^{\ln 2} e^{2x} - 4e^{x} + 4 \, dx$$

$$= \left[\frac{e^{2x}}{2} - 4e^{x} + 4x\right]_0^{\ln 2}$$

$$= \left(\frac{e^{2\ln 2}}{2} - 4e^{\ln 2} + 4\ln 2\right) - \left(\frac{e^{20}}{2} - 4e^{0} + 0\right)$$

$$= 2 - 8 + 4\ln 2 - \frac{1}{2} + 4 + 0$$

$$= 4\ln 2 - \frac{5}{2}$$

b) i) $V = 2500\left(1 - \frac{t}{50}\right)^2$

for $t = 10$

$$V = 2500\left(1 - \frac{10}{50}\right)^2$$

$$= 2500 \times \frac{16}{25}$$

$$= 1600 \text{ L}$$

ii) $$\frac{dV}{dt} = 2 \times 2500\left(1 - \frac{t}{50}\right) \times -\frac{1}{50}$$

$$= -100\left(1 - \frac{t}{50}\right)$$

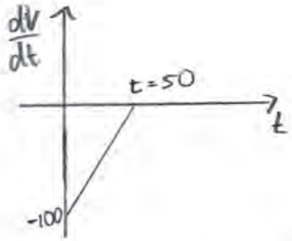
$$= -100 + 2t$$

when $t = 20$ $\quad \frac{dV}{dt} = -100 + 2 \times 20$

$$= -60 \text{ L/min}$$

Question 7 continued

b) iii) $\frac{dV}{dt} = -100 + 2t$



The fastest rate at which water leaves the tank is a $t=0$ ie: $\frac{dV}{dt} = -100 L/min$

c) 31 DEC 2011 $(A_1 = 7000 \times 1.05)$

31 DEC 2012 $(A_2 = 7000 \times 1.05^2 + 7000 \times 1.05)$

31 DEC 2013 $(A_3 = 7000 \times 1.05^3 + 7000 \times 1.05^2 + 7000 \times 1.05)$

i) $A_3 = 7000\ (1.05 + 1.05^2 + 1.05^3)$

ii) Assume $A_n = 7000\ (1.05 + 1.05^2 + \ldots + 1.05^n)$

ie Geometric Series $a = 1.05$
$r = 1.05$
$n = 31$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$S_{31} = \frac{1.05 \times (1.05^{31} - 1)}{0.05}$$

$$= 74.2988\ldots$$

$\therefore A_{31} = \$7000 \times S_{31}$
$= \$520\ 091.81$

Question 8

a) i) $\sqrt{x} = \frac{1}{\sqrt{x}}$ $\therefore$ P is the point $(1,1)$

$1 = x^2$

$\therefore x = 1$

$y = \sqrt{1}$
$= 1$

ii) Area $= \int_0^1 y^2\,dy + \int_1^2 y^{-2}\,dy$

note: $y = \sqrt{x}$
$\underline{\underline{x = y^2}}$
$y = \frac{1}{\sqrt{x}}$
$\underline{\underline{x = y^{-2}}}$

$= \left[\frac{y^3}{3}\right]_0^1 + \left[\frac{y^{-1}}{-1}\right]_1^2$

$= \left(\frac{1}{3} - 0\right) + \left(-\frac{1}{2} + 1\right)$

$= \frac{5}{6}$ units^2

b) $\log_7 x - \log_7 4 = 2\log_7 3$

$\log_7\left(\frac{x}{4}\right) = \log_7 9$

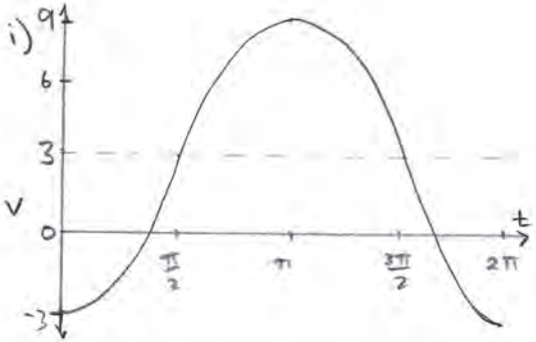
$\frac{x}{4} = 9$

$x = 36$

Question 8 continued

c) $v = 3 - 6\cos t$

i)

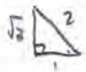


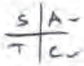
ii) Particle is at rest when $v = 0$

$0 = 3 - 6\cos t$

$6\cos t = 3$

$\cos t = \frac{1}{2}$





$\therefore t = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3}$

$= \frac{\pi}{3}$ or $\frac{5\pi}{3}$

iii) $a = \frac{d(v)}{dt}$

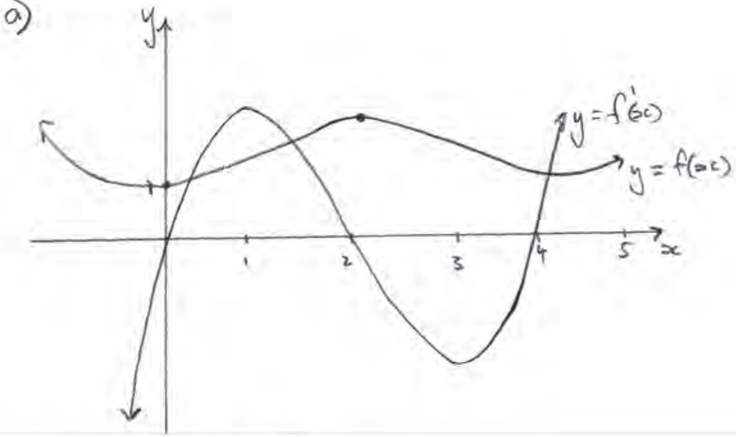
$\therefore a = 6\sin t$

iv) $\sin t$ has its first maximum when $t = \frac{\pi}{2}$

$\therefore$ The particle first reaches its maximum acceleration when $t = \frac{\pi}{2}$ seconds.

Question 9

a)



b) i) $M = M_0 e^{-kt}$

$\frac{dM}{dt} = M_0 \times -k \times e^{-kt}$

$= -k \times M_0 e^{-kt}$

$= -kM$

ii) when $t = 6$ $\frac{M}{M_0} = \frac{1}{2}$

$\frac{1}{2} = e^{-k \times 6}$

$\ln \frac{1}{2} = -6k$

$k = \frac{\ln 2}{6}$

iii) $M = 1$, $k = \frac{\ln 2}{6}$, $t = 15.6$

$1 = M_0 \times e^{-\frac{\ln 2}{6} \times 15.6}$

$= 6.06286\ldots$

$\therefore$ Approx 6kg was shipped

Question 9 continued

c) i) $P = 2k + 2k\theta + \pi k$

$= k(2\theta + \pi + 2)$

ii) Area = Area of sector + Area of semicircle

$= \frac{1}{2}r^2\theta + \frac{1}{2}\pi R^2$ $\quad r = 2k$

$R = k$

$= \frac{1}{2}(2k)^2\theta + \frac{1}{2}\pi k^2$

$1 = 2k^2\theta + \frac{1}{2}\pi k^2$

$2k^2\theta = 1 - \frac{1}{2}\pi k^2$

$\theta = \dfrac{1 - \frac{1}{2}\pi k^2}{2k^2}$

$= \dfrac{1}{2k^2} - \dfrac{\pi}{4}$

iii) for $\theta = \dfrac{1}{2k^2} - \dfrac{\pi}{4}$

$P = k(2\theta + \pi + 2)$

$= k\left(2\left(\frac{1}{2k^2} - \frac{\pi}{4}\right) + \pi + 2\right)$

$= k\left(\frac{1}{k^2} - \frac{\pi}{2} + \pi + 2\right)$

$= \dfrac{1}{k} + \dfrac{k\pi}{2} + 2k$

$= \dfrac{1}{k} + k\left(2 + \dfrac{\pi}{2}\right)$

Question 10

a) i) $S = kd^2w$

$\therefore S = k(27 - w^2)w$

$= k(27w - w^3)$

$d^2 + w^2 = 27$

$d^2 = 27 - w^2$

ii) $\dfrac{dS}{dw} = k(27 - 3w^2)$

$\dfrac{dS}{dw} = 0$ for maximum strength

$0 = k(27 - 3w^2)$

$w^2 = 9$

$w = \pm 3$

$= 3$ (ignore -ve dimension)

when $w = 3$, $d^2 = 27 - 9$

$= 18$

$d = \sqrt{18}$

Check max $\dfrac{d^2S}{dw^2} = -6wk$

< 0 (assume $w > 0$)

($k > 0$ given)

$\therefore$ max strength when

$w = 3$cm + $d = \sqrt{18}$ cm

iii) $2w^2 = 27$

$w^2 = \dfrac{27}{2}$

$\therefore d = w = \sqrt{\dfrac{27}{2}}$

Question 10 continued

a)iii) $S_{square} = kd^2 w$

$= k \times \frac{27}{2} \times \sqrt{\frac{27}{2}}$

$S_{max} = kd^2 w$

$= k \times 18 \times 3$

$$\frac{S_{max}}{S_{square}} = \frac{54k}{\frac{27k}{2} \times \sqrt{\frac{27}{2}}}$$

$$= \frac{4}{\sqrt{\frac{27}{2}}}$$

$$= 1.08866\ldots$$

$\therefore$ $S_{max} > 1.08 \times S_{square}$ as Required.

b) i) $f(x) = x\left((\ln x)^2 - 2\ln x + 2\right)$

$u = x$

$u' = 1$

$v = (\ln x)^2 - 2\ln x + 2$

$v' = \frac{2\ln x}{x} - \frac{2}{x}$

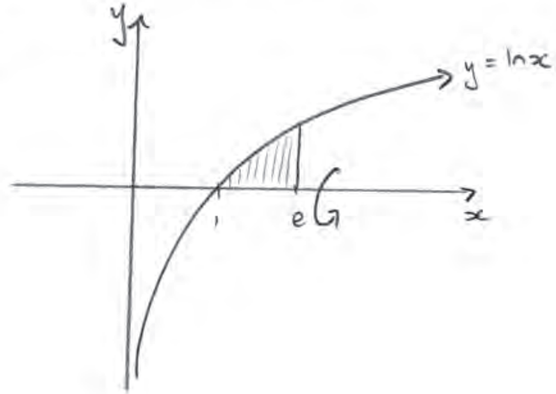
$f'(x) = x\left(\frac{2\ln x}{x} - \frac{2}{x}\right) + \left((\ln x)^2 - 2\ln x + 2\right)$

$= \cancel{2\ln x} \cancel{-2} + (\ln x)^2 \cancel{- 2\ln x} \cancel{+2}$

$= (\ln x)^2$

Question 10 continued

b) ii)



$$V = \pi \int_1^e y^2 \, dx$$

$$= \pi \int_1^e (\ln x)^2 \, dx$$

$$= \pi \left[x\left((\ln x)^2 - 2\ln x + 2\right) \right]_1^e$$

$$= \pi \left[e(1^2 - 2 + 2) - 1(0^2 - 0 + 2) \right]$$

$$= \pi (e - 2) \text{ units}^3$$